

The Maslov gerbe

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Abstract

Let $\text{Lag}(E)$ be the grassmannian of lagrangian subspaces of a complex symplectic vector space E . We construct a Maslov class which generates the second integral cohomology of $\text{Lag}(E)$, and we show that its mod 2 reduction is the characteristic class of a flat gerbe with structure group \mathbb{Z}_2 . We explain the relation of this gerbe to the well-known flat Maslov line bundle with structure group \mathbb{Z}_4 over the real lagrangian grassmannian, whose characteristic class is the mod 4 reduction of the real Maslov class.

1 Introduction

In real symplectic geometry, the **Maslov class** is a canonical class $\mu(E)$ which generates the first integral cohomology of the lagrangian grassmannian $\text{Lag}(E)$ of a symplectic vector space E . Given any $L_0 \in \text{Lag}(E)$, the homology class dual to $\mu(E)$ is represented by the **Maslov cycle** of E with respect to L_0 ; the latter is the cooriented, codimension 1, real algebraic variety Σ_{L_0} (with singularities of codimension 3 in $\text{Lag}(E)$) consisting of the lagrangian subspaces in E which are not transverse to E . The mod 4 reduction of $\mu(E)$ has the further interpretation as the characteristic class (or holonomy) of the **Maslov line bundle**, a flat $U(1)$ bundle over $\text{Lag}(E)$. Although the full Maslov class is essential in symplectic topology, only the mod 4 reduction matters in microlocal analysis, where the Maslov class was first discovered. (See [8]. The topological description as a cohomology class was first given by Arnol'd [1].)

When E is the sum $V \oplus \bar{V}$ of a symplectic vector space V and the same vector space with the opposite symplectic structure, $\text{Lag}(E)$ contains the graphs of the symplectic automorphisms of V , so the symplectic group $Sp(V)$ embeds

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in $\text{Lag}(E)$. The Maslov class on $\text{Lag}(E)$ pulls back to twice the generator of $H^1(Sp(V), \mathbb{Z})$, so a (4-valued) parallel section of the Maslov line bundle of E restricts to $Sp(V)$ as a double covering which can be identified with the metaplectic group $Mp(V)$.

This linear algebra and topology is applied to microlocal analysis as follows [7][8]. Over a cotangent bundle T^*X , the tangent spaces to the fibres form a lagrangian subbundle L_0 of the symplectic vector bundle $T(T^*X)$, i.e. a section of the bundle $\text{Lag}(T(T^*X))$ of lagrangian subspaces of the tangent spaces to T^*X . The Maslov cycles attached to the individual tangent spaces then combine to form a cycle in $\text{Lag}(T(T^*X))$. This cycle is dual to a canonical Maslov class in $H^1(\text{Lag}(T(T^*X)), \mathbb{Z})$, whose mod 4 reduction is again the characteristic class of a flat $U(1)$ bundle, the Maslov bundle $\mathcal{M}_X \rightarrow \text{Lag}(T(T^*X))$.

For each lagrangian immersion $j : \Lambda \rightarrow T^*X$, there is a natural ‘‘Gauss map’’ $G_j : \Lambda \rightarrow \text{Lag}(T(T^*X))$, and the ‘‘Maslov canonical operator’’ (or cousins going by various names) produces (sometimes distributional, or depending on a ‘‘quantization’’ parameter) sections of the bundle $\sqrt{|\bigwedge^{\text{top}} T^*X|}$ of half densities on X from sections of $\sqrt{|\bigwedge^{\text{top}} T^*\Lambda|} \otimes G_j^*(\mathcal{M}_X)$. In other words, the Maslov bundle is a kind of ‘‘transition object’’ from the half densities on lagrangian submanifolds of T^*X to those on X itself.

What becomes of all this when the real numbers are replaced by the complex numbers? In this paper, we will answer this question at the level of linear algebra and topology, leaving applications to microlocal analysis for the future.

$\text{Lag}(E)$ is now a complex manifold, and the Maslov cycle associated to L_0 is a complex subvariety of complex codimension 1, with singularities of complex codimension 3. Repeating in the complex setting the proof of Theorem 3.4.9 in [3], one can show that, as in the real case, the complement of the Maslov cycle is contractible, and (since a complex subvariety is coorientable) that this cycle is therefore dual to a generator of $H^2(\text{Lag}(E), \mathbb{Z})$ which is independent of the choice of L_0 . We call this generator the **complex Maslov class**. The main point of this paper is that the mod 2 reduction of the complex Maslov class is the characteristic class of a flat $U(1)$ *gerbe* over $\text{Lag}(E)$ which we naturally call the **Maslov gerbe**. Objects of this gerbe are local square roots of the unitarized determinant bundle associated to the tautological vector bundle over $\text{Lag}(E)$. The absence of global objects of this gerbe is well known and corresponds to the nonexistence of various other objects, including half-forms on general complex manifolds and a connected double covering of $Sp(2n, \mathbb{C})$. The latter impossibility plays a central role in recent work of Omori et al [9] on star-exponentials of complex quadratic polynomials, which was in fact the stimulus for the present work.

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2 Square roots of line bundles as \mathbb{Z}_2 gerbes

Let λ be a complex line bundle over a manifold M . Following Brylinski [2], we may define a \mathbb{Z}_2 gerbe $\sqrt{\lambda}$ over M whose nontriviality is the obstruction to the existence of a (tensor) square root of λ . Namely, we apply Proposition 5.2.3 of [2] to the exact sequence of groups

$$0 \rightarrow \mathbb{Z}_2 \rightarrow GL(1) \xrightarrow{s} GL(1) \rightarrow 0,$$

where s is the squaring homomorphism. (We will identify \mathbb{Z}_k with the multiplicative group of k 'th roots of unity.) In the case of a hermitian line bundle, everything is essentially the same, with the sequence above replaced by

$$0 \rightarrow \mathbb{Z}_2 \rightarrow U(1) \xrightarrow{s} U(1) \rightarrow 0.$$

According to the cited proposition, there is associated to the line bundle λ a \mathbb{Z}_2 gerbe which represents the obstruction to lifting the bundle through s . A construction of Giraud (Theorem 5.2.8 in [2]) associates to this gerbe a characteristic class $\gamma(\sqrt{\lambda}) \in H^2(M, \mathbb{Z}_2)$. Finally, Theorem 5.2.9 in [2] asserts that this characteristic class is obtained from the characteristic class of λ in sheaf cohomology by the connecting homomorphism of one of the exact sequences above. Comparing these with the exponential sequences usually associated with line bundles,

$$0 \rightarrow \mathbb{Z} \rightarrow \mathbb{C} \rightarrow GL(1) \rightarrow 0,$$

or

$$0 \rightarrow \mathbb{Z} \rightarrow \mathbb{R} \rightarrow U(1) \rightarrow 0$$

in the hermitian case, we conclude that the Giraud characteristic class is simply the mod 2 reduction of the Chern class $c(\lambda)$.

These constructions may be described in more concrete terms. For each open $\mathcal{U} \subseteq M$, $\sqrt{\lambda}(\mathcal{U})$ is the groupoid whose objects are pairs (τ, ι) consisting of a (hermitian in the $U(1)$ case) line bundle τ and an isomorphism ι from the tensor square τ^2 to the restriction $\lambda|_{\mathcal{U}}$. A morphism from (τ, ι) to (τ', ι') is a bundle isomorphism $\sigma : \tau \rightarrow \tau'$ such that $\iota' \sigma^2 \iota^{-1}$ is the identity automorphism of $\lambda|_{\mathcal{U}}$, where σ^2 is the tensor square of σ . It is easy to see that any two objects in $\sqrt{\lambda}(\mathcal{U})$ are isomorphic and that the automorphism group of (τ, ι) may be identified with the continuous (hence locally constant) functions on \mathcal{U} with values in \mathbb{Z}_2 . Thus, $\sqrt{\lambda}$ is a gerbe with band \mathbb{Z}_2 .

To determine the Giraud characteristic class of a gerbe \mathcal{G} with discrete structure group, we choose a good covering of M by open subsets \mathcal{U}_i which are the domains of objects \mathcal{O}_i . On $\mathcal{U}_i \cap \mathcal{U}_j$, we choose an isomorphism σ_{ij} from the

restriction of \mathcal{O}_j to that of \mathcal{O}_i . On a triple intersection $\mathcal{U}_i \cap \mathcal{U}_j \cap \mathcal{U}_k$, the composition $\gamma_{ijk} = \sigma_{ij}\sigma_{jk}\sigma_{ki}$ is a constant function with values in the structure group; these functions define a Čech 2-cocycle which is, up to coboundaries, independent of choices. We denote the resulting cohomology class by $\gamma(\mathcal{G})$.

To relate the Giraud class of $\sqrt{\lambda}$ to the Chern class of λ , we determine the Chern class by choosing a good covering of M by open subsets \mathcal{U}_i and trivializations $\epsilon_i : \lambda|_{\mathcal{U}_i} \rightarrow \mathcal{U}_i \times \mathbb{C}$. On $\mathcal{U}_i \cap \mathcal{U}_j$, the composition $\epsilon_i \epsilon_j^{-1}$ is represented by a $GL(1)$ -valued function r_{ij} . We choose complex-valued (real-valued in the hermitian case) functions θ_{ij} such that $r_{ij} = e^{2\pi i \theta_{ij}}$. On $\mathcal{U}_i \cap \mathcal{U}_j \cap \mathcal{U}_k$, $r_{ij} r_{jk} r_{ki} = 1$, so $c_{ijk} = \theta_{ij} + \theta_{jk} + \theta_{ki}$ is a constant \mathbb{Z} -valued function. The integers c_{ijk} form a cocycle which represents the Chern class $c(\lambda) \in H^2(M, \mathbb{Z})$.

We now compute the Giraud class by using the same covering and letting $\tau_i = \mathcal{U}_i \times \mathbb{C}$, with $\iota_i((m, z) \otimes (m, z)) = \epsilon_i^{-1}(m, z^2)$. Choosing $\sigma_{ij}(m, z) = (m, e^{\pi i \theta_{ij}(m)} z)$ gives

$$\gamma_{ijk} = \sigma_{ij}\sigma_{jk}\sigma_{ki} = e^{\pi i(\theta_{ij} + \theta_{jk} + \theta_{ki})} = e^{\pi i c_{ijk}} = (-1)^{c_{ijk}}.$$

Thus we have proved:

Proposition 2.1 *If λ is a complex line bundle over M , then the Giraud class $\gamma(\sqrt{\lambda})$ is the mod 2 reduction of the Chern class $c(\lambda)$.*

3 Reduction modulo 2 of the complex Maslov class

In this section, we will identify a line bundle λ over $\text{Lag}(E)$ whose Chern class is the complex Maslov class when E is a complex symplectic vector space.

First, we note that, just as we can form tensor powers of line bundles, we can also pass in a canonical way from a line bundle λ to a unitary line bundle $\arg(\lambda)$ which we will call the **unitarization** of λ . It is just the bundle associated to the principal bundle of frames of λ via the homomorphism $\arg : z \mapsto z/|z|$ from $GL(1, \mathbb{C})$ to $U(1)$. It is the tensor quotient of λ by the line bundle $|\lambda|$ obtained via the absolute value homomorphism from $GL(1, \mathbb{C})$ to the positive real numbers \mathbb{R}_+ . (When λ is a real line bundle, we work with its complexification.) Since \mathbb{R}_+ is contractible, $|\lambda|$ is always trivial, and hence the characteristic (Chern or Stiefel-Whitney) class of $\arg(\lambda)$ is the same as that of λ .

Now let η be the tautological vector bundle over $\text{Lag}(E)$, i.e. the bundle whose fibre over L is L itself. Fix a lagrangian subspace L_0 and a nonzero element ϕ of $\bigwedge^{\text{top}}(E/L_0)^*$. The pullbacks of ϕ to the elements of $\text{Lag}(E)$ by the projection $E \rightarrow E/L_0$ define a holomorphic section of the line bundle $\bigwedge^{\text{top}} \eta^*$ which vanishes precisely along the Maslov cycle Σ_{L_0} ; it is transverse to the zero section along the regular part of Σ_{L_0} , where $L \cap L_0$ is 1-dimensional. (Thus, the Maslov cycle is a divisor associated to this line bundle.) It follows immediately that the complex Maslov class is the Chern class of the line bundle $\bigwedge^{\text{top}} \eta^*$ and hence that of $\arg(\bigwedge^{\text{top}} \eta^*)$ as well. Applying Proposition 2.1, we arrive at

Proposition 3.1 *Let E be a (finite-dimensional) complex symplectic vector space. The mod 2 reduction of the complex Maslov class is the Giraud characteristic class of the \mathbb{Z}_2 gerbes $\sqrt{\bigwedge^{\text{top}} \eta^*}$ and $\sqrt{\arg(\bigwedge^{\text{top}} \eta^*)}$, where η is the tautological bundle over the lagrangian grassmannian $\text{Lag}(E)$, and $\arg(\)$ is the unitarization.*

Remark 3.2 We can say a bit more about the complex Maslov class. The conormal bundle to the Maslov cycle along the smooth locus is the complex line bundle whose fibre over each element L consists of the quadratic forms on the line $L \cap L_0$. Being the square of a line bundle, this conormal bundle has a Chern class which vanishes mod 2. But this Chern class is dual to the self-intersection of the Maslov cycle with itself, and so is equal to the square of the Maslov class. Were it not for the singularities of the Maslov cycle, we could conclude immediately that the cup square (in degree 4) of the complex Maslov class is always an even class (i.e. zero modulo 2) or, equivalently, that the square of the (mod 2) Giraud class is zero. In fact, Ralph Cohen (private communication) has verified this conclusion by representing $\text{Lag}(\mathbb{C}^{2n})$ as the homogeneous space $Sp(n)/U(n)$. The vanishing of the square of the complex Maslov class is also consistent with results in Section 6 of [4] to the effect that the even \mathbb{Z}_2 cohomology rings of certain complex manifolds are isomorphic to the \mathbb{Z}_2 cohomology rings of their real forms, via an isomorphism which halves degrees.

4 Reduction modulo 4 of the real Maslov class

In preparation for our comparison of the real and complex Maslov classes, we give a geometric description of the Maslov bundle in the real case as a square root of the flat \mathbb{Z}_2 bundle $\arg(\bigwedge^{\text{top}} \eta^*)$.

Since the trivial bundle $|\bigwedge^{\text{top}} \eta^*|$ has a natural “positive” square root, the bundle $\sqrt{|\bigwedge^{\text{top}} \eta^*|}$ of half-densities, this is equivalent to studying square roots of $\bigwedge^{\text{top}} \eta^*$, which are generally known as bundles of half-forms. (See, for instance, [5]).

We look first at the case where E is 2-dimensional, so that the lagrangian grassmannian is simply a projective line, and $\bigwedge^{\text{top}} \eta^*$ is just η^* itself. In terms of canonical coordinates (q, p) on E , we have two coordinate systems on $\text{Lag}(E)$. The first assigns to each line of the form $p = aq$ its slope a ; the second assigns to each line of the form $q = bp$ its inverse slope b . The range of each coordinate system is an entire (real or complex) line, and the transition map, defined where a and b are nonzero, is $b = 1/a$. Bases for η^* on the two neighborhoods are given by dq and dp , with the transition relation $dp = adq$. Bases for the bundle $\sqrt{|\eta^*|}$ of half-densities are then given by $\sqrt{|dq|}$ and $\sqrt{|dp|}$, with the transition relation $\sqrt{|dp|} = \sqrt{|a|}\sqrt{|dq|}$.

Constructing a square root of $\arg(\eta^*)$ is equivalent to choosing a square root $\sqrt{\arg(a)}$ on the set where $a \neq 0$. This can be done in two inequivalent ways,

in the real case. For $a > 0$, we may without loss of generality take the positive square root 1, while for $a < 0$ we may take either i or $-i$. Once we have made this choice, we take bases for $\sqrt{\arg(\eta^*)}$ which we may call $\sqrt{\arg(dq)}$ and $\sqrt{\arg(dp)}$, with the transition relation $\sqrt{\arg(dp)} = \sqrt{\arg(a)} \sqrt{\arg(dq)}$.

Now let us follow a parallel section of $\sqrt{\arg(\eta^*)}$ around a loop in $\text{Lag}(\mathbb{R}^2)$ which starts at the line $a = 1$ and rotates the line in the counterclockwise direction in the (q, p) plane. As a increases to infinity and the line becomes vertical, we may take $\sqrt{\arg(dq)}$ as this section. We write this section as $\sqrt{\arg(a)}^{-1} \sqrt{\arg(dp)} = \sqrt{\arg(dp)}$ and then follow it as the line crosses the p -axis into the region of negative a . To cross the horizontal axis and return to the line $a = 1$, we first return to the q representation by writing $\sqrt{\arg(dp)} = \sqrt{\arg(a)} \sqrt{\arg(dq)} = \pm i \sqrt{\arg(dq)}$. We have thus returned to $\pm i$ times our original section, and the holonomy around our loop is multiplication by $e^{\pm i\pi/2}$. The sign depends on our choice of square root, which can be chosen so that this holonomy is given by the mod 4 reduction of the Maslov class. We will not concern ourselves here with the choice of sign, which depends on the sign convention used in the definition of the Maslov class, via the choice of coorientation of the Maslov cycle.

When the dimension of E is greater than 2, we can write E as a direct sum of a symplectic plane and another symplectic summand, and take a loop in $\text{Lag}(E)$ consisting of the direct sum of a line in the symplectic plane, moving as in the paragraph above, with a fixed lagrangian subspace in the second summand. This reduces the general case to that where E is 2-dimensional, and we arrive at the following result.

Proposition 4.1 *Let E be a (finite-dimensional) real symplectic vector space. the mod 4 reduction of the real Maslov class is the characteristic class of a (flat) \mathbb{Z}_4 bundle whose tensor square is $\arg(\bigwedge^{\text{top}} \eta^*)$, where η is the tautological bundle over $\text{Lag}(E)$.*

5 Relation between the real and complex Maslov classes and their reductions

We will begin with the following general result.

Proposition 5.1 *Let \mathcal{G} be a flat $U(1)$ gerbe over the sphere \mathbb{CP}^1 . Let \mathcal{O}_+ and \mathcal{O}_- be objects of \mathcal{G} defined on neighborhoods \mathcal{U}_+ and \mathcal{U}_- of the northern and southern hemispheres respectively. Then the value of the Giraud class of \mathcal{G} on the fundamental cycle of \mathbb{CP}^1 is equal to the holonomy of the flat $U(1)$ bundle $\mathcal{O}_+ \otimes \mathcal{O}_-^{-1}$ around the equator \mathbb{RP}^1 .*

Proof. We use the concrete construction of the Giraud class introduced in Section 2. \mathbb{CP}^1 may be triangulated as the boundary of a 3-simplex by four triangles, one of which exactly fills the southern hemisphere, while the other

three have a common vertex at the north pole and are numbered in counter-clockwise order when viewed from above. We enlarge these triangles to open subsets $\mathcal{U}_0 \dots \mathcal{U}_3$ which form a nice covering, and where $\mathcal{U}_0 = \mathcal{U}_-$, while the other three subsets are contained in \mathcal{U}_+ . We may then choose the object \mathcal{O}_0 to be \mathcal{O}_i , and the other three to be the restrictions of \mathcal{O}_+ .

When i and j are between 1 and 3, we choose the isomorphism σ_{ij} over $\mathcal{U}_i \cap \mathcal{U}_j$ to be the one induced from the identifications with \mathcal{O}_+ . We choose the isomorphisms σ_{0i} to be parallel sections of the flat bundles $\mathcal{O}_0 \otimes \mathcal{O}_i^{-1}$, all of which are identified with $\mathcal{O}_- \otimes \mathcal{O}_+^{-1}$. The value of the Giraud cocycle on the fundamental class of the sphere is the product in $U(1)$ of the four compositions (constant sections of trivial bundles) $\gamma_{ijk} = \sigma_{ij}\sigma_{jk}\sigma_{ki}$ where (ijk) takes the values (123), (032), (013), and (021). Now $\sigma_{123} = 1$ by construction, and, suppressing the natural isomorphisms, we find the remaining three values to be $\sigma_{03}\sigma_{02}^{-1}$, $\sigma_{01}\sigma_{03}^{-1}$, and $\sigma_{02}\sigma_{01}^{-1}$. But the product of these three “jumps” is exactly the inverse of the holonomy of the flat bundle $\mathcal{O}_- \otimes \mathcal{O}_+^{-1}$ around the equator $\mathbb{R}P^1$ oriented as the boundary of the northern hemisphere, i.e. with the usual orientation on \mathbb{R} when the point at infinity is ignored. Thus, it is equal to the holonomy of the inverse bundle $\mathcal{O}_+ \otimes \mathcal{O}_-^{-1}$.

□

We apply the proposition above to the gerbe $\sqrt{\bigwedge^{\text{top}} \eta^*}$ over $\text{Lag}(E)$, where E is complex. The general case can be reduced to that where E is 2-dimensional, so we may identify $\text{Lag}(E)$ with $\mathbb{C}P^1$ as we have done before. Objects \mathcal{O}_\pm on the northern and southern hemispheres must restrict to the two different square roots of $\bigwedge^{\text{top}} \eta^*$ on the equator. (If they were the same, there would be a global object.) Therefore, the tensor quotient $\mathcal{O}_+ \otimes \mathcal{O}_-^{-1}$ is the tensor square of one of these square roots, i.e. of the Maslov line bundle or its inverse. It is for this reason (we might say that it is because the sphere is made of *two* hemispheres), that the structure group $\mathbb{Z}_2 \subset U(1)$ of the Maslov gerbe on the complex Lagrangian grassmannian is the square of the structure group $\mathbb{Z}_4 \subset U(1)$ of the Maslov line bundle on the real lagrangian grassmannian.

Remark 5.2 There is a similar phenomenon in “one degree lower.” In [7], the holonomy in \mathbb{Z}_4 of the Maslov line bundle is effectively expressed as the product of an even number of jumps in \mathbb{Z}_8 (see the remark on page 163).

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